ON ISOCHRONOUS SUSPENSION OF CLOCK PENDULA

(OB IZOKHRONNOM PODVESE MAIATNIKA V CHASAKH)

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In this paper we investigate the influence of an isochronous suspension, which was used for astronomical clocks by Fedchenko (AChF) [1] on the period of pendulum oscillation. The formula which is obtained for the period allows one to estimate the influence of changes in the suspension parameters on the isochronous property.

1. As is well known, the period of oscillation of a pendulum is given by the formula

$$T = T_0\left(1 + \frac{\theta_0^2}{16} + \ldots\right)$$

where θ_0 is the angular amplitude and T_0 is a quantity which is independent of θ_0 (the period to the zeroth order of approximation). Anisochronism, i.e. the dependence of the period on the amplitude, is one of the most important sources of inaccuracies in pendulum clocks. This insufficiency was overcome to a certain extent by Fedchenko [1] whose clocks differ from existing astronomical clocks in their simplicity and high degree of accuracy. This was realized through the use of a new type of pendulum suspension – the so-called isochronous suspension. The suspension consists of three plane springs placed at the position of equilibrium of the pendulum in a single plane, perpendicular to the plane of oscillation of the pendulum. The upper ends of the springs are embedded in the body of the clock, while the lower ends are rigidly attached to the pendulum arm (Fig. 1). The two short side springs are completely identical and their symmetrical location guarantees the absence of transverse bending during the pendulum oscillation; for the subsequent consideration they may be replaced by a single spring of double thickness. The middle spring is longer than the side springs, and the points of support of its upper and lower ends are correspondingly displaced above and below the points of support of the ends of the short springs. The isochronous action of the clock is regulated by an upward or downward shift of the long spring along the longitudinal axis of the pendulum.

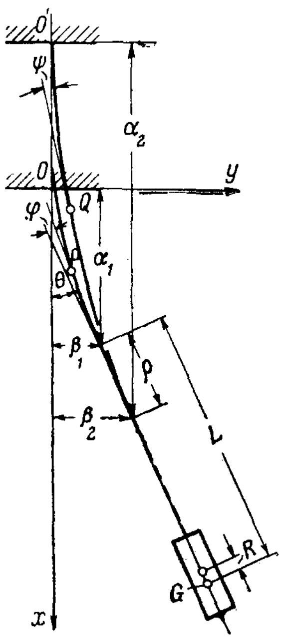

Fig. 1.

We note in passing that although all the springs are thickened at their ends we shall consider their transverse cross-sections to be constant, and shall take as the lengths of the springs the lengths of their thin portions.

We take as the origin of coordinates the point of support to the body of the clock of the upper end of the short spring, while as independent variables we take the coordinates of the lower end α_1 and β_1 and the angle of inclination θ (Fig. 1). Neglecting the kinetic energy of the suspension, whose mass is much smaller than the mass m of the pendulum, it is easy to write the Lagrangian function for the system

$$L = \tfrac{1}{2} m(L^2 + R^2)\,\dot{\theta}^2 + \tfrac{1}{2} m\,(\dot{\alpha}_1^2 + \dot{\beta}_1^2) + mL\dot{\theta}\,(\dot{\beta}_1 \cos\theta - \dot{\alpha}_1 \sin\theta) - mg\,(l_1 - \alpha_1) - mgL\,(1 - \cos\theta) - U_1 - U_2$$

Here L is the distance from the point of support of the short spring at the pendulum arm to the center of gravity G, R is the radius of gyration, chosen so that the moment of inertia with respect to G is equal to mR^2, l_1 is the length of the short spring, U_1 and U_2 are the energies of deformation of the short and long springs of the suspension, respectively.

Denoting by X_i, Y_i and N_i (in units of weight of the pendulum) the forces and moments which act on the pendulum from the sides of the short ($i = 1$) and long ($i = 2$) springs, we obtain the equations of motion of the pendulum:

$$\begin{gathered}\ddot{\alpha}_1 - L\ddot{\theta}\sin\theta - L\dot{\theta}^2\cos\theta = g\,(1 - X_1 - X_2)\\ \ddot{\beta}_1 + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta = -g\,(Y_1 + Y_2)\\ R^2/g\cdot\ddot{\theta} + N_1 + N_2 = L\,(Y_1\cos\theta - X_1\sin\theta) + (L - \rho)\,(Y_2\cos\theta - X_2\sin\theta)\end{gathered} \qquad (1.1)$$

Here ρ is the displacement of the lower end of the long spring along the pendulum rod. Having obtained X_i, Y_i and N_i from a consideration of the spring deformations, one could attempt to obtain an expression for the period of change of θ from (1.1). However, within the accuracy which interests us, the period may be obtained by the use of a method which is similar to that which was worked out in [2] for the ordinary single-spring suspension.

2. In the design under consideration $R \ll L$, and hence we may restrict ourselves to an investigation of the mathematical pendulum ($R = 0$). The third of Equations (1.1) is transformed into an equation coupling the variables. Moreover, $l_1 \ll L$, and hence we shall assume that the change in α_1 is determined only by the angle of inclination, thereby excluding from consideration the vibration of the point of attachment of the pendulum to the suspension. Of the three variables, only one remains independent. We choose for this θ.

Assuming that the amplitude of oscillation of an astronomical clock is ordinarily 1-3°, that is, $\theta \ll 1$, we solve the problem by the method of successive approximations. To the zeroth order, the system (1.1) is

$$\begin{gathered} X_1^\circ + X_2^\circ = 1 \\ \ddot{\beta}_1 + L\ddot{\theta} = -g\,(Y_1^\circ + Y_2^\circ) \\ N_1^\circ + N_2^\circ = L\,(Y_1^\circ - X_1^\circ\theta) + (L - \rho)\,(Y_2^\circ - X_2^\circ\theta) \end{gathered} \tag{2.1}$$

where the index ° denotes terms of the corresponding order in θ in the expressions for X_i, Y_i and N_i.

It is obvious that the X_i are even functions of θ and that the X_i° are constants which should be equal to the corresponding values in the position of equilibrium. Here the springs are unbent but are extended by a certain amount, so that the length of the short spring is l_1 and that of the long spring l_2. The first relation (2.1) and the condition of equilibrium of extension of the springs

$$\Delta l = \frac{X_i^\circ}{K_i} \qquad \left(K_i = \frac{E_i S_i}{l_i mg}\right)$$

(where E_i, S_i are the Young's moduli and the cross-sectional areas of the springs, respectively ($i = 1.2$)) allows one to write

$$X_i^\circ = \frac{K_i}{K_1 + K_2}, \qquad \frac{X_1^\circ}{X_2^\circ} = \frac{E_1 S_1 l_2}{E_2 S_2 l_1}$$

For the suspension which is used in the AChF clocks $E_1 = E_2$, $S_1 \approx S_2$ and $l_2 \gg l_1$, that is, $X_1^\circ \gg X_2^\circ$.

In the approximation under consideration, we may assume that the spring lengths do not change and determine Y_i and N_i from the equations for pure bending. For the short spring (Fig. 1) this has the form (for example, [3], Part II, (18.1), (18.6) and (19.3))

$$E_1 I_1 \frac{d^2\varphi}{dp_2} = X_1 \sin\varphi - Y_1 \cos\varphi, \qquad I_1 = \frac{b_1 h_1^3}{12} \tag{2.2}$$

Here b_1 and h_1 are the width and thickness of the spring; $0 << p << l_1$, $0 << \phi << \theta$, where $p = 0P$ is the distance from 0 to the generic point P and ϕ is the angle of inclination of the tangent with the x-axis at the point P. At the ends of the spring the following conditions must be satisfied:

$$\varphi(0) = 0, \quad \varphi(l_1) = \theta, \quad \beta_1 = \int_0^{l_1} \sin\varphi\, dp, \quad \alpha_1 = \int_0^{l_1} \cos\varphi\, dp \tag{2.3}$$

In the case at hand $\phi << \theta << 1$, and to the zeroth order approximation (2.2) is

$$\frac{1}{\omega_1^2}\frac{d^2\varphi^\circ}{dp^2} = \varphi^\circ - y_1^\circ, \qquad \omega_1^2 = \frac{X_1^\circ}{E_1 I_1}, \qquad y_1^\circ = \frac{Y_1^\circ}{X_1^\circ} \tag{2.4}$$

Correspondingly, from (2.3) we have the boundary conditions

$$\varphi^\circ(0) = 0, \quad \varphi^\circ(l_1) = \theta, \quad \beta_1 = \int_0^{l_1} \varphi^\circ\, dp, \quad \alpha_1^\circ = l_1 \tag{2.5}$$

With the aid of (2.5), the solution of (2.4) has the form

$$\varphi^\circ = \frac{\sinh \omega_1 p}{\sinh 2u_1}\theta + y_1^\circ\left[1 - \frac{\cosh(\omega_1 p - u_1)}{\cosh u_1}\right] \tag{2.6}$$

where

$$2u_1 \doteq \sqrt{\frac{12\, mgl_1^2 l_2}{h_1^2(E_1S_1l_2 + E_2S_2l_1)}}, \qquad y_1^\circ = \frac{\omega_1\beta_1 - \theta\tanh u_1}{2u_1 - 2\tanh u_1} \tag{2.7}$$

Introducing the distance from 0′ (Fig. 1) to the generic point Q, $q = 0'Q$ for the long spring, and denoting the angle of the tangent with the x-axis at the point Q by ψ, it is easy to obtain an equation, together with its solution, which is analogous to (2.2) through (2.7), respectively. In order to obtain ϕ° and ψ° in final form, we use the third of relations (2.1). As is well known (for example, [3])

$$N_1^\circ = E_1 I_1 \left(\frac{d\varphi^\circ}{dp}\right)_{p=l_1} \tag{2.8}$$

The quantity N_2° is determined analogously. Using $\beta_2 = \beta_1 + \rho\theta$ (Fig.1) in the approximation under consideration, we obtain $\beta_1 = \lambda\theta$, where, under the conditions $L >> l_2 >> l_1$, $X_1^\circ >> X_2^\circ$, $2u_2 > 1$ and $2u_1 >> 1$, which usually obtain

$$\lambda = l_1\left(1-\frac{\tanh u_1}{2u_1}\right)\left[1+\frac{S_2 E_2 l_1}{S_1 E_1 l_2}\,\frac{2u_1-2\tanh u_1}{2u_2-2\tanh u_2}\times\right.$$
$$\left.\times\frac{(l_2-l_1-\rho)(2u_2/l_2)+(l_1/l_2)(u_2/u_1)\tanh u_1-\tanh u_2}{2u_1-\operatorname{th} u_1}\right] \qquad (2.9)$$

Substituting β_1 into (2.7), we have

$$y_1{}^\circ = \frac{(2u_1/l_1)\,\lambda-\tanh u_1}{2u_1-2\tanh u_1}\,\theta = W_1\theta \qquad (2.10)$$

Then $\phi^\circ = \kappa\,\theta$, where

$$\varkappa = \frac{\sinh \omega_1 p}{\sinh 2u_1} + W_1\left[1-\frac{\cosh(\omega_1 p-u_1)}{\cosh u_1}\right] \qquad (2.11)$$

In an analogous manner, for the long spring $y_2{}^\circ = W_2\theta$ and $\psi^\circ = \nu\,\theta$, where W_2 and ν differ from W_1 and κ by a change of index from 1 to 2 and p to q.

The second of Equations (2.1) transforms into

$$(L+\lambda)\,\ddot\theta = -g\,(W_1X_1{}^\circ + W_2X_2{}^\circ)$$

Hence, the zeroth-order period is

$$T_0 = 2\pi\sqrt{\frac{L-\lambda}{g\,(W_1X_1{}^\circ+W_2X_2{}^\circ)}} \qquad (2.12)$$

Under the restrictions used in the derivation of (2.9), it can be shown that $W_1X_1{}^\circ + W_2X_2{}^\circ = 1$, hence

$$T_0 = 2\pi\sqrt{\frac{L+\lambda}{g}}$$

For $E_2 \to 0$, $u_2 \to \infty$

$$T_0 = 2\pi\sqrt{(1/g)\,[L+l_1(1-\tanh u_1/2u_1)]}$$

which coincides with the result obtained in [2] for the ordinary suspension.

3. To see the effect of the suspension on the anisochronous oscillations of the pendulum, we examine the following (first) approximation. In this approximation, the system of equations (1.1) has the form

$$(1/g)\,[-\ddot\alpha_1 + L(\theta\ddot\theta+\dot\theta^2)] = X_1{}^1 + X_2{}^1$$
$$\ddot\beta_1 + L\ddot\theta - L(\ddot\theta\theta^2/2+\theta\dot\theta^2) = -g\,(Y_1{}^\circ+Y_2{}^\circ+Y_1{}^1+Y_2{}^1) \qquad (3.1)$$
$$(N_1{}^\circ+N_1{}^1)+(N_2{}^\circ+N_2{}^1) = (1-\theta^2/2)\,[L(Y_1{}^\circ+Y_1{}^1)+(L-\rho)(Y_2{}^\circ+Y_2{}^1)] +$$
$$+(\theta-\theta^3/6)\,[L(X_1{}^\circ+X_1{}^1)+(L-\rho)(X_2{}^\circ+X_1{}^1)]$$

Here, the upper index 1 denotes terms in X_i, Y_i and N_i of the corresponding order with respect to θ.

Assuming that the X_i^1 are known and that the change of the spring lengths is insignificant, we find Y_i^1 and N_i^1, respectively, from the equations of bending of the springs. To do this, we set $\phi = \phi^\circ + \phi^1$ in Equation (2.2); then for the determination of ϕ^1 we have

$$\frac{1}{\omega_1^2}\frac{d^2\varphi^1}{dp^2} = \varphi^1 - y_1^1 - \gamma(p) \tag{3.2}$$

where

$$y_1^1 = \frac{Y_1^1}{X_1^\circ}, \qquad \gamma(p) = W_1\frac{\varkappa^2}{2}\theta^3 - \frac{\varkappa^3}{6}\theta^3 + \varkappa\frac{X_1^1}{X_1^\circ}\theta$$

With the notation $\alpha_1 = \alpha_1^\circ + \alpha_1^1$, we obtain from (2.3) and (2.5) the boundary conditions

$$\varphi^1(0) = 0, \quad \varphi^1(l_1) = 0, \quad \int_0^{l_1}\left(\varphi^1 - \frac{1}{6}\varkappa^3\theta^3\right)dp = 0, \quad \alpha_1^1 = -\frac{\theta^2}{2}\int_0^{l_1}\varkappa^2\,dp \tag{3.3}$$

Solving (3.2) by the method of variation of parameters and using (3.3), we find

$$y_1^1 = \frac{\omega_1}{2u_1 - 2\tanh u_1}\int_0^{l_1}\left\{\gamma(p)\left[1 - \frac{\cosh(\omega_1 p - u_1)}{\cosh u_1}\right] + \frac{\varkappa^3}{6}\theta^3\right\}dp$$

It is clear that

$$n_1^1 = \frac{1}{\omega_1^2}\left(\frac{d\varphi^1}{dp}\right)_{p=l_1}, \qquad n_1^1 = \frac{N_1^1}{X_1^\circ} \quad \text{or} \quad n_1^1 = \int_0^{l_1}\gamma(p)\frac{\sinh \omega_1 p}{\tanh 2u_1}dp - y_1^1\frac{\tanh u_1}{\omega_1}$$

By entirely the same method one may find ψ^1, y_2^1 and n_2^1.

Using the first and third of relations (3.1), we express β_1 in terms of θ to the order of θ^3. Substituting the expression which is obtained into (2.7) and then into the second of Equations (3.1), we bring it to the form

$$(L+\lambda)\ddot{\theta} + g(W_1X_1^\circ + W_2X_2^\circ)\theta = F(\theta) \tag{3.4}$$

$$\begin{aligned} F(\theta) = &-\frac{g}{(L+\lambda)^2}\Big\{(X_1^1J_1^{(2)} + X_2^1J_2^{(2)} - X_2^1\rho)\theta - \frac{L\theta^3}{6} - \\ &-\frac{\theta^3}{6}(X_1^\circ J_1^{(4)} + X_2^\circ J_2^{(4)}) + \rho\theta^3X_2^\circ\left(\frac{1}{6} - \frac{2}{3}W_2\right) + \\ &+\frac{\theta^3}{3}(W_1X_1^\circ J_1^{(3)} + W_2X_2^\circ J_2^{(3)}) + \frac{2}{3}L\theta^3\frac{W_1X_1^\circ + W_2X_2^\circ}{L+\lambda}(\lambda + J_1^{(2)})\Big\} \end{aligned} \tag{3.5}$$

where

$$J_1^{(n)} = \int_0^{l_1} \varkappa^n \, dp, \qquad J_2^{(n)} = \int_0^{l_2} \nu^n \, dq$$

For the determination of the X_i^1 we introduce the quantities

$$\sigma = (\theta^2 / 2)\, a, \qquad a = J_1^{(2)} + \rho - J_2^{(2)}$$

Expression (3.3) for a_1^1 shows that σ characterizes the difference between the elevation of the end of the short spring together with the portion of the arm of length ρ and the end of the long spring if they were raised independently, and if the forces X_i acting on them were constant and equal to X_1°. (For approximate estimates we note that by replacing the bent springs by straight lines it is easy to obtain $a = (l_2 - l_1 - \rho)(l_1 + \rho)/l_2$.) In reality, the spring ends are attached to the pendulum arm and cannot rise independently; and depending on the design of the suspension two cases are possible.

1) The case $\sigma > 0$; that is, the end of the long spring should be additionally raised or the end of the short spring lowered. If the bending energy is much smaller than the change in energy of the initial extension of the springs, then it can be assumed that the necessary matching of the position of the lower ends of the springs is attained only at the expense of their contraction or extension. Inasmuch as in our case $K_1 \gg K_2$, the contraction of the long spring plays a basic role, while a smaller one is played by the extension of the short spring. It may be supposed that the force that acts on the long spring is changed to

$$X_2^1 = -K_2 \sigma (1 - K_2 / K_1) \tag{3.6}$$

2) The case $\sigma < 0$; for $K_1 \gg K_2$ it is basically necessary to lower the end of the long spring. This can be accomplished not only at the expense of its extension, but also at the expense of the form of the bent spring, and in this case it is necessary to say what sort of deformation plays the larger role.

We return to the first case (as will be seen it is expressly this case that occurs). From the first of Equations (3.1) it follows that $|X_1^1 + X_2^1| \sim \theta^2$, while $|X_2^1| \sim \theta^2 (l_2 - l_1 - \rho)/\Delta l$; but usually $\Delta l \ll l_2 - l_1 - \rho$, hence $|X_1^1 + X_2^1| \ll |X_2^1|$. This allows one to assume that $X_1^1 = -X_2^1$. In addition, $\kappa \leqslant 1$ and $\nu \leqslant 1$, that is, $J_1^{(n)} \leqslant l_1$ and $J_2^{(n)} \leqslant l_2$, and also $X_i^\circ \leqslant 1$, $|Y_i| \sim 1$. Hence, all of the terms in (3.5) are much smaller than the second and the first, whose denominators contain small quantities Δl, and all except the second and first may be discarded:

$$F(\theta) = \frac{g\theta^3}{(L+\lambda)^2}\left[\frac{L}{6} - \frac{K_2 a^2}{2}\left(1 - \frac{K_2}{K_1}\right)\right]$$

The solution of equations with small nonlinearities of the type (3.4) has been worked out, for example, in [4]. In the present case, the correction to the period is obtained as

$$\frac{\Delta T}{T_0} = \frac{\theta_0^2}{16}\left[1 - \frac{3E_2S_2}{Lmgl_2}\left(1 - \frac{S_2E_2l_1}{S_1E_1l_2}\right)(J_1^{(2)} + \rho - J_2^{(2)})^2\right] \qquad (3.7)$$

where θ_0 is the angular amplitude. Using (2.11), we obtain

$$J_1^{(2)} = \frac{1}{2}\left\{3W_1\lambda - \left(\frac{1}{\sinh 2u_1} + W_1 \tanh u_1\right)^2 l_1 + \frac{1}{\omega_1}(\coth 2u_1 + W_1 \tanh u_1)\right\} \qquad (3.8)$$

and analogously $J_2^{(2)}$.

Formula (3.7) is easily obtained from an examination of the energy of the system. Upon deviation, the potential energy of the ordinary pendulum of length L is increased by approximately $mgL(1 - \cos\theta)$, that is, not proportional to θ^2 but somewhat slower. This leads to the presence of nonharmonic terms in the equations of motion and to anisochronous oscillations. Upon the inclination of the pendulum with the isochronous suspension, there occurs a redistribution of firces acting on the side of each of the springs and on the pendulum. If $\sigma > 0$, then the force acting on the side of the long spring is decreased when the pendulum is inclined, corresponding to a decrease in the initial energy of extension $\Delta U_2 = -1/8K_2mga^2\theta^4$. At the same time, the extensional energy of the short spring $\Delta U_1 = 1/8(K_2^2/K_1)mga^2\theta^4$ is increased. For $K_1 \gg K_2$ the energy of the entire suspension is decreased. The entire system (suspension and pendulum) is conservative, hence the decrease in the potential energy of the suspension corresponds to an increase in the energy of the pendulum.

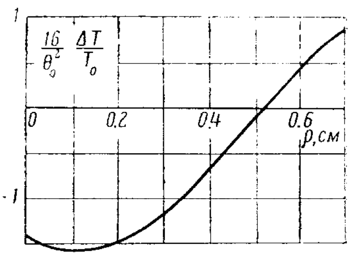

Fig. 2.

The term $1/24\ mgL\theta_4$ in the expression for the potential energy of the pendulum gives the correction to the period $(\Delta T)/T_0 = 1/16\ \theta^2$, and in the present case it is simple to obtain a formula which coincides with (3.7).

4. We examine (3.7), using numerical data that apply to the suspension of AChF clocks: $E_1 = E_2 = 2.1 \times 10^6$ kg/cm^2, $l_1 = 0.3$ cm, $l_2 = 1.6$ cm, $b_1 = 0.8$ cm, $b_2 = 0.6$ cm, $h_1 = 1 \times 10^{-2}$ cm, $h_2 = 1.5 \times 10^{-2}$ cm; and, in addition, $L = 100$ cm, $mg = 10$ kg.

From (2.7), (2.9), (2.10) and (3.8), and likewise from analogous expressions for the long spring, one can compute the change in the isochronous property of the suspension as a function of the displacement ρ of the lower end of the long spring along the pendulum arm. The results of such computation are shown on Fig. 2. For $\rho = \rho_0 = 0.5$ cm, which corresponds to $a = 0.19$ cm, complete isochronism of the oscillation is obtained and $\Delta T = 0$. For $\rho > 0.7$ cm $\sigma = 0$, and the above investigation is not valid.

It should be remarked that Expression (3.6) is meaningful only when the change in the energy of compression (extension) of the springs in the suspension is much larger than the change in the bending energy. In [3, Part II, Sect. 20] it is shown that this condition is violated for $\beta_2 \ll h_2$ (for the long spring); for $\rho = 0.5$ cm, this corresponds to $\theta_1 \ll 1°$. On the other hand, for a certain angle $\theta = \theta_2$, the length of the long spring becomes equal to its length before extension, and for $\theta \geqslant \theta_2$ the change in the energy of the support is determined only by the bending of the spring. It is seen that the forces that are excited in this case are much smaller than those that are obtained from (3.6), and the isochronous property of the suspension should no longer obtain.

From the condition $\Delta l = \sigma$ it is easy to obtain

$$\theta_2{}^2 = \frac{2}{a\,(K_1 + K_2)}$$

which for $\rho = 0.5$ cm is about $3°$.

Formula (4.1) obtained above allows one to estimate the influence of a change of parameters over a region of angles in which the oscillations are isochronous. For $K_1 \gg K_2$, $\theta_2{}^2 = (2l_1mg)/(aES_1)$. If the change in the suspension parameters does not violate the relationship between K_1 and K_2, then it is clear from this expression that an increase in l_1 or a decrease in S_1 leads to an increase in the region of isochronism. However, this is true only for small changes in these quantities, when it may be assumed that a is constant.

In spite of a number of assumptions made in the solution and indicated above, an analysis of (3.7) leads to several conclusions.

1) The isochronous property of the suspension is both a result of the extension (compression) and bending of the springs. In this, the isochronous suspension is different from the usual single-spring suspension [2], where the extension of the spring hardly changes and where the investigation may be restricted to bending alone. In the isochronous suspension a basic role is played by the additional deformation of the long spring, which is associated with the incompatibility between the points

of support of its ends and the ends of the short spring [1]. The dependence in (3.7) on the weight of the pendulum, the arm length L, the transverse cross-section of the long spring S_2 and its Young's modulus E_2 agrees with experimental data. For $E_2 = 0$ the isochronous property of the support vanishes.

2) The estimate which has been made shows that the oscillations can be isochronous only to angles of the order of 3°. In the experiments, isochronous oscillations were observed from 30′ to 40′ up to 2.5° to 3°, which agrees with the estimate.

3) It is seen from the above graph that $\Delta T(\theta) = 0$ for $\rho = 0.5$ cm. In actual suspensions ρ can be adjusted, which regulates the degree of isochronism. Complete isochronism was actually observed for $\rho_0 = 0.5$ to 0.6 cm.

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